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Gromov-Witten Invariants via Algebraic Geometry

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Abstract

Calculations of the number of curves on a Calabi-Yau manifold via an instanton expansion do not always agree with what one would expect naively. It is explained how to account for continuous families of instantons via deformation theory and excess intersection theory. The essential role played by degenerate instantons is also explained.

1 Introduction

In recent years, there has been much interaction between string theory and algebraic geometry. In particular, the Yukawa couplings for the heterotic string compactified on a Calabi-Yau manifold X can be calculated using mirror symmetry [1]. The result can be interpreted as an instanton expansion on X to give the numbers of rational curves on X . Similar results hold for higher genus [2, 3].

Certain puzzles arise from this analysis. In examples, the effective number of curves, or Gromov-Witten invariants, can have strange behaviors. These numbers can be negative [4, 5, 6, 7, 8], nonintegral [2, 3], or have positive contributions when it is geometrically clear that there are no curves of the type naively predicted [4, 5, 6, 7, 8].

There are two techniques that are needed to resolve these puzzles. The first is better known. In many situations, continuous families of instantons

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can develop. The path integral will require an integration over these instanton moduli spaces. This can be done in one of two ways. The fermion zero modes can be calculated and then integrated over as well. This tends to be rather difficult. The more successful approach is to find a bundle whose Euler class calculates the virtual fundamental class of the moduli space in question [9]. In practice, this may be accomplished via deformation theory [10, 11] and excess intersection theory [12]. This gives rise to an obstruction analysis and consequently an effective finite contribution of continuous families, as illustrated for example in [5, 13], in the appendix to [2], or using symplectic geometry [14]. The last method is based on the rigorous mathematical definition of Gromov-Witten invariants [15, 16]. The second technique is prompted by the realization that certain degenerate instantons are required in the path integral. There are several approaches to degenerate instantons, e.g the appendix to [2] and [13]; emphasis will be given here to the use of stable maps as in the latter reference.

The combination of these two techniques resolves all puzzles currently posed by calculations on the mirror theory. There are still many calculations that cannot presently be carried out, but to the extent that the calculations can be performed, the results agree with those predicted by mirror symmetry. Furthermore, it is now completely clear in which situations negative and nonintegral contributions are possible, and these kinds of invariants are not observed in any other cases. Finally, in all examples for which there are contributions to the instanton expansions yet no corresponding curves, it is now apparent that there *are* degenerate instantons in these cases which have a right to contribute.

In Section 2 generalities are given about counting curves and how excess intersection theory can be used to handle continuous families of instantons. Several approaches to degenerate instantons are outlined in Section 3. Several types of examples are given in Section 4: constant maps in Section 4.1, families of smooth curves in Section 4.2, genus 0 multiple covers in Section 4.3, and degree 1 maps to curves of lower genus in Section 4.4.

2 Counting Curves

2.1 Generalities

Consider the heterotic string theory compactified on the Calabi-Yau target space X . There are well known quantities in string theory which are corrected by genus g world sheet instantons [17, 2, 3], and through an instanton expansion give methods for computing number of genus g curves on X (or more precisely, the corresponding Gromov-Witten invariant). To set up notation, let $\mathcal{K}(X) \subset H^2(X, \mathbf{C})$ denote the Kähler cone of X . Assume that $\mathcal{K}(X)$ is simplicial, generated by classes H_1, \dots, H_n , where $n = \dim H^2(X) = h^{1,1}(X)$ is the number of Kähler parameters. This assumption is valid if X is an appropriate hypersurface or complete intersection in a toric variety associated to a reflexive polyhedron [18, 19].

It is desired to geometrically understand the quantity which is naively written as ²

$$F_g = \sum_{C_g \subset X} q_1^{C_g \cdot H_1} \dots q_n^{C_g \cdot H_n} \quad (1)$$

where the sum is over genus g (possibly degenerate) holomorphic instantons.

Here $\omega = B + iJ$ is the complexified Kähler class (B field and metric), and in terms of the chosen basis for $\mathcal{K}(X)$, put

$$\omega = \sum_{i=1}^n t_i H_i q_i = e^{2\pi i t_i} \quad (2)$$

The formula (1) for F_g must be understood to contain terms associated to *degenerate instantons*, including what is often called the classical term (which results from integration over the space of constant maps from the world sheet).

Typically, F_g is calculated via mirror symmetry, and then geometric information can be inferred from the result inductively using the quantities $F_{g'}$ for $g' < g$ and an understanding of certain degenerate instantons.

The quantity F_0 is just the prepotential. The quantity F_1 is introduced in [2], while F_g for $g \geq 2$ is introduced in [3].

Put $\delta_i = q_i \frac{\partial}{\partial q_i}$. Then the third derivative $\delta_i \delta_j \delta_k F_0$ is a Yukawa coupling,

²This is not standard notation, but the notation has been chosen to emphasize the similarities between the various genera.

which is calculated to be³

$$\sum_{C \subset X} (C \cdot H_i)(C \cdot H_J)(C \cdot H_k) q_1^{C \cdot H_1} \dots q_n^{C \cdot H_n} \quad (3)$$

where the sum is over genus 0 instantons.

For ease of notation, if $[C] = \gamma \in H^2(X, \mathbf{Z})$, put

$$q^\gamma = \prod_{i=1}^n q_i^{C \cdot H_i} \quad (4)$$

Further, put $d_i = C \cdot H_i$ whenever the meaning of C is clear from context. Let n_γ be the number of genus 0 curves C with $[C] = \gamma$. The expected dimension of the space of genus 0 curves (or any genus for that matter) is 0, so it makes sense to seek the number of such curves, counted appropriately. Then $\delta_i \delta_j \delta_k F_0$ can be rewritten as

$$\sum_{\gamma} n_{\gamma} d_i d_j d_k q^{\gamma} + \dots = \sum_{f: P^1 \rightarrow X, f(p_{\alpha}) \in H_{\alpha}} q^{\gamma} + \dots, \quad (5)$$

where the $p_{\alpha} \in \mathbf{P}^1$ are marked points and the dots denote terms arising from degenerate instantons. It is thus seen that n_{γ} is (essentially) a Gromov-Witten invariant.⁴

Similarly, the quantity $\delta_i F_1$ allows for enumeration of genus 1 curves, and is seen to be

$$\sum_{\gamma} e_{\gamma} d_i q^{\gamma} + \dots = \sum_{f: E \rightarrow X, f(p_i) \in H_i} q^{\gamma} + \dots \quad (6)$$

where E is any elliptic curve and e_{γ} is the number of genus 1 instantons. This will again be corrected for degenerate instantons later.

The mathematical reason why it is necessary to consider 3 point functions and 1 point functions in (5) and (6) above is that the space of automorphisms of a genus g curve has respective dimensions 3 and 1 for $g = 0$ and 1, so rigidification is necessary to get finite numbers of curves.

³This is still a naive formula; the role of degenerate instantons will be clarified later.

⁴There are several different proposed definitions of the Gromov-Witten invariant; it is expected that they all agree.

2.2 Excess Intersection

It is not possible in general to get finite instanton moduli spaces, even after rigidification as above in theories with generic parameter values.

To see what to do, first consider a special case. Suppose that the instanton moduli space is infinite yet smooth of finite dimension for a special parameter value. Suppose that the moduli space becomes finite for general parameter values.

Deformation theory [10, 11] gives rise to an obstruction bundle on the instanton moduli space. As the special parameter value gets smoothly perturbed to a general parameter value, a section of the obstruction bundle results, and the limits of the finite moduli spaces are recognizable as the zero locus of this section of the obstruction bundle. The number of zeros is just the degree of the top Chern class (or Euler class) of the obstruction bundle.

Notice that the form of the result is independent of the assumption that the moduli space becomes finite for general parameter values. So it is natural to make the hypothesis that the effective finite contribution of our moduli space is just the degree of the top Chern class of the obstruction bundle.

In string theory, the calculation is performed by an integration over the fermion zero modes; it is being asserted that the Chern class calculation achieves the same result.

This assertion can be justified even in situations where the moduli space is infinite for generic parameter values. The idea is to interpret our calculation in a more general context, so that what would be called general parameter values in the original context becomes special in the more general context. For example, the calculation of the Yukawa couplings can be understood as an A-model calculation. The A-model makes sense for almost complex manifolds [20]. The moduli space of rigidified pseudoholomorphic curves on a generic almost complex manifold is finite, and the assertion is reduced to the case already checked. An example is given in [5].

There is a more general method which will merely be mentioned here. The above procedure is a special case of *excess intersection theory* [12]. In excess intersection theory, there are not only bundles but a certain subcone of these bundles, the *normal cone*. In fact, a calculation in [9] is recognizable as excess intersection theory. This led the author in [21] to define an analogous *virtual normal cone* in the special case that the reduced moduli space is smooth. The resulting effective finite contribution was called the *equivalence* following the

corresponding terminology in excess intersection theory [12]. Other authors refer to the same object as the *virtual fundamental class*. J. Li and G. Tian [22] have recently found a good definition of the virtual normal cone in the general case. Work is in progress to compare this definition with the existing definition [15, 16] using symplectic geometry.

3 Degenerate Instantons

Degenerate instantons first appeared in [23] in the course of compactifying the instanton moduli space of multiple cover maps to a rational curve. On the other hand, the fact that the instantons were degenerate played no role in the calculation.

It became clear from [2] that degenerate instantons play a fundamental role; the path integral can reduce to a space on which all instantons are degenerate. This occurs for degenerate maps of degree 1 from an elliptic curve to a rational curve.

One approach is to identify instantons $f : C \rightarrow X$ with their graphs $\Gamma \subset C \times X$, and then consider degenerate configurations of Γ which do not necessarily arise from maps f . This is the approach taken in [23] and the appendix to [2]. This space of degenerate instantons is a subset of the *Hilbert scheme* [24] of $C \times X$, (or a relative Hilbert scheme, if C can vary).

Another approach, and the one that will be expanded on in the sequel, is to consider stable maps as the degenerate instantons.

One should ask whether these methods of compactification are compatible. From the point of view of the path integral, it is not a complete surprise that different methods of compactifying the instanton moduli space give the same answers. We can view the definition of the path integral as an integral over the interior of the moduli space of all maps. If part of the path integral is concentrated near the boundary of the moduli space of all maps, then the method of compactification should not affect the result of the calculation, just the way that the calculation proceeds.

On the other hand, from the viewpoint of deformation theory, a very precise method of calculation has been proposed which ignores the interior of the moduli space, so there is no apparent reason why the methods are equivalent. Yet there are no discrepancies evident to date between different methods.

Kontsevich introduced the notion of a stable map as follows [13].

Definition *A stable map $f : C \rightarrow X$ consists of the following data.*

- *A curve $C = C_1 \cup \dots \cup C_n$ whose only singularities are ordinary double points.*
- *Marked points $p_1, \dots, p_r \in C$ in the smooth locus of C .*
- *A holomorphic map $f : C \rightarrow X$ such that there are no continuous automorphisms of f which are the identity on p_1, \dots, p_r .*

The last condition means that smooth components C_i of genus 0 (resp. 1) have at least 3 (resp. 1) special (i.e. nodal or marked) points.

For counting curves, it is necessary to take

$$r = \begin{cases} 0 & g \geq 2 \\ 1 & g = 1 \\ 3 & g = 0 \end{cases} \quad (7)$$

This choice has been explained in the last paragraph of Section 4.

One early impetus to the study of degenerate instantons arose in conversations between the author and D. Morrison in 1991. It was realized that during a change of complex structure of the quintic threefold $\mathbf{P}^4[5]$, a conic can approach a line pair. Suppose that the line pair occurs for $t = 0$, while smooth conics occur for $t \neq 0$. For $t \neq 0$ we have maps from \mathbf{P}^1 with $\gamma \cdot H = 2$ (H =hyperplane class), but no such maps for $t = 0$. Several strategies for including degenerate instantons were proposed to explain this discontinuity.

Stable maps provide the most straightforward resolution of this problem. More generally, suppose that a smooth rational curve D degenerates to a union $D_1 \cup D_2$ of two smooth rational curves, meeting transversally at a point.

Now count rigidified stable maps whose image is D , and compare to those whose image is $D_1 \cup D_2$. The contribution of D to a Yukawa coupling $\langle H_1 H_2 H_3 \rangle$ is

$$(D \cdot H_1)(D \cdot H_2)(D \cdot H_3) \quad (8)$$

For $D_1 \cup D_2$ there are several types of contributions. The source C of the stable map must be a union $C_1 \cup C_2$ of two \mathbf{P}^1 s. The three marked points

can be distributed arbitrarily between the curves C_i . Then each of the curves C_i can map to either D_1 or D_2 . Including all possibilities, the calculation becomes

$$\prod_i (D_1 \cdot H_i) + \prod_i (D_2 \cdot H_i) + \sum_{\{a,b,c\}=\{1,2,3\}} [(D_1 \cdot H_a)(D_1 \cdot H_b)(D_2 \cdot H_c) + (D_1 \cdot H_a)(D_2 \cdot H_b)(D_2 \cdot H_c)] \quad (9)$$

Using the relation

$$D \cdot H_i = (D_1 + D_2) \cdot H_i, \quad (10)$$

it is clear that the results of (8) and (9) agree.

4 Examples

In this section, examples will be given for the use of stable maps as they arise in the instanton expansion for F_g .

4.1 Constant Maps

The first application is to the classical term, which corresponds to constant maps.

4.1.1 Genus 0

In this case, 3 marked points are needed. For a constant map with 3 marked points to be stable, the source curve C must be irreducible, hence just \mathbf{P}^1 . More generally, a constant map is stable if and only if the source of the map is a stable curve. The three marked points may be assumed to be $0, 1, \infty \in \mathbf{P}^1$. The only other data needed to describe the map is the image point $p \in X$. Thus the moduli space is isomorphic to X . The dimension is 3, so the obstruction bundle is trivial. The point condition given by H_i translates into the cohomology class of H_i on X . This gives [25]

$$\delta_i \delta_j \delta_k F_0 = H_i H_j H_k + \dots \quad (11)$$

4.1.2 Genus 1

A single marked point $p \in C$ is needed in this case. As noted above, the source of the map is a stable curve. Pointed stable curves $p \in C$ are parameterized by $\overline{M}_{1,1}$. Including the target point $p \in X$ gives the moduli space $\overline{M}_{1,1} \times X$. This space has dimension 4, hence there is a rank 3 obstruction bundle whose Chern class must be calculated. For fixed C , the obstruction space is given by the second hypercohomology group \mathbf{H}^2 of the complex⁵

$$T_C(-p) \rightarrow f^*T_X \quad (12)$$

where $T_C(-p)$ denotes the sheaf of vector fields on C which vanish at p [13]. The resulting cohomology sequence shows that the obstruction space is just $H^1(f^*T_X)$. Since $f^*(T_X)$ is a trivial bundle, this gives $\mathbf{H}^2 \simeq H^1(\mathcal{O}_X^3)$. This globalizes to the rank 3 bundle

$$\mathcal{H}^* \otimes T_X, \quad (13)$$

a bundle on $\overline{M}_{1,1} \times X$. Here \mathcal{H} is the Hodge bundle of holomorphic 1 forms on C . The Hodge bundle arises because of the isomorphism $H^1(\mathcal{O}_C) \simeq (H^{1,0}(C))^*$.

Inserting the condition $p \in H_i$ gives

$$\begin{aligned} \int_{\overline{M}_{1,1} \times X} c_3(\mathcal{H}^* \otimes T_X) \cdot H_i &= \\ [\int_X c_2(T_X) \cdot H_i] \int_{\overline{M}_{1,1}} c_1(\mathcal{H}^*) &= \\ -\frac{1}{12} \int_X c_2(T_X) \cdot H_i & \end{aligned} \quad (14)$$

the constant term of $\delta_i F_1$ [2]

4.1.3 Genus $g \geq 2$

In this case, there are no marked points, and the parameter space is $\overline{M}_g \times X$ which has dimension $3g$.

If $f : C \rightarrow p \in X$ is a constant map, then the obstruction space is again seen to be $\mathcal{H}^* \otimes T_X$, a rank $3g$ bundle on $\overline{M}_g \times X$. So the $\gamma = 0$ contribution to F_g is

$$\int_{\overline{M}_g \times X} c_{3g}(\mathcal{H}^* \otimes T_X) = (-1)^g \frac{1}{2} \chi(X) \int_{\overline{M}_g} c_{g-1}(\mathcal{H})^3 \quad (15)$$

agreeing with [3] up to sign.

⁵This calculation is only valid when C is irreducible, but it can be shown that the result is the same in any case.

4.2 Families of Smooth Curves

Suppose that X contains a family of smooth curves with smooth parameter space B . Denoting the total space of the family by \mathcal{C} , there results a diagram

$$\begin{array}{ccc} \mathcal{C} & \subset & B \times X \\ \pi \downarrow & & \downarrow \\ B & = & B \end{array} \quad (16)$$

The fibers $\pi^{-1}(b) = C_b \subset \{b\} \times X$ are identified with curves $C_b \subset X$. Since the curves are smooth, the point conditions may be inserted at the outset of the analysis for genus $g < 2$. This reduces the deformation problem from the study of maps to the study of embedded curves. Let N_b denote the normal bundle of $C_b \subset X$. Then

$$\begin{array}{ll} H^0(N) & = T_b B \\ H^1(N) & = \text{obstruction space} \end{array} \quad (17)$$

Note that $H^1(N)$ is dual to $H^0(N)$ via the pairing

$$H^1(N) \otimes H^0(N) \rightarrow H^1(\wedge^2 N) \simeq H^1(T_C^*) = \mathbf{C} \quad (18)$$

so the obstruction bundle is globally T_B^* .

It follows that the contribution to the Gromov-Witten invariant is

$$c_{\dim(B)}(T_B^*) = (-1)^{\dim(B)} \chi(B) \quad (19)$$

Examples.

1. For a ruled surface $E \subset X$:

$$\begin{array}{ccc} E & \subset & X \\ \pi \downarrow & & \\ B & & \end{array} \quad (20)$$

we get the contribution $2g - 2$, where g is the genus of the parameter curve B .

Note that if $g(B) \geq 1$, the complex structure on X is not generic: deforming the complex structure of X destroys E , replacing it with $2g - 2$ isolated rational curves [4].

2. If $\mathbf{P}^2 \subset X$, then lines in \mathbf{P}^2 are parameterized by a \mathbf{P}^2 , hence the contribution to the Gromov-Witten invariant is 3. Plane conics are parameterized by \mathbf{P}^5 , hence the contribution to the Gromov-Witten invariant is -6 . In fact, a deformation of almost complex structure reveals a finite number of pseudoholomorphic oriented curves, with signed number -6 [14, 5].

Without exception, these check out against calculations performed using mirror symmetry [4, 5, 6, 7, 8].

4.3 Genus 0 Multiple Covers

Degree k multiple covers of a smooth curve $C \subset X$ with $[C] = \gamma$ contribute $d_i d_j d_k q^\gamma$ to $\delta_i \delta_j \delta_k F_0$ [1, 23]. As an alternative approach, the space of instantons can be compactified in two ways: using the normal bundle of graphs of maps or stable maps. Using the normal bundle of maps gives rise to the same moduli space as in [23] but a different obstruction bundle. The result is that all answers agree, giving for $\delta_i \delta_j \delta_k F_0$

$$H_i H_j H_k + \sum_{\gamma} \frac{d_i d_j d_k n_{\gamma} q^{\gamma}}{1 - q^{\gamma}} \quad (21)$$

The use of stable maps explains a phenomenon which was not previously understood. In all cases where there is a map $\mathbf{P}^1 \rightarrow X$ which is not an embedding, there are contributions to $n_{k\gamma}$ which arise in addition to the usual contributions for multiple covers. Up until now, this was only realized by the calculation of Yukawa couplings via mirror symmetry.

Example: Consider a nodal rational curve D . Such a curve is the image of an immersion $g : \mathbf{P}^1 \rightarrow X$ which maps two points of \mathbf{P}^1 to the node p and is otherwise an embedding. The curve D has two branches near p . The multiple covers discussed above arise in this situation by composing g with a multiple cover $\mathbf{P}^1 \rightarrow \mathbf{P}^1$.

In addition, other k -fold covers of D arise as follows. Let $C = C_1 \cup \dots \cup C_k$ be a union of k copies of \mathbf{P}^1 meeting transversely, with $C_i \cap C_{i+1} = p_i$ for $i = 1, \dots, k-1$. Form a map $f : C \rightarrow D$ by setting the restriction of f to each C_i to be the immersion g above. In addition, it is required that $f(p_i) = p$ for all i , and then further required that for each i , the branches of D to which complex disks around p_i in C_i and C_{i+1} are mapped are not the same branch. This phenomenon occurs in many examples, e.g. [5, 6, 7, 8].

Example: Union of a line L and a conic D inside a \mathbf{P}^2 contained in X . Here there are two intersection points q_1, q_2 with two branches at each point. For simplicity, rather than talking about the homology class γ , the planar degree will be described instead.

The construction of multiple covers is similar to that above. Multiple cover maps do not come in arbitrary degree; the degree can only be added to in multiples of 3. For example, to get a degree 4 degenerate instanton from a degree 1 instanton: start with an instanton $f : C_1 \rightarrow L$. Put $C = C_1 \cup C_2 \cup C_3$ where the C_i are rational curves and $C_i \cap C_{i+1} = p_i$ as before. Extend f to $\tilde{f} : C \rightarrow L \cup D$ by reparameterizing f as necessary so that $f(p_1) = q_1$ (or q_2 , which is similar), then define \tilde{f} on C_2 so that \tilde{f} is an isomorphism of C_2 onto D with $\tilde{f}(p_2) = q_2$, and finally defining \tilde{f} to be an isomorphism of C_3 onto L .

4.4 Degree 1 maps to curves of lower genus

This is the “bubbling phenomenon” of [26, 27, 2, 3]. Consider the contribution $e_{g,g'}$ of genus g' curves C to F_g with $g' < g$. For ease of notation, put $g'' = g - g'$. These degenerate instantons arise from stable maps $f : C_1 \cup C_2 \rightarrow C$ with C_1, C_2 of respective genera g'', g' such that $f(C_1)$ is a point $p \in C$ and f is an isomorphism from C_2 to C . Restrict attention to the case $g \geq 2$ to avoid having to say anything about the point conditions (these do not create any new problems). The moduli space of such stable maps is

$$C \times \overline{M}_{g'',1}, \quad \dim = 3g'' - 1 \quad (22)$$

It is very desirable to work out the obstruction theory for this deformation problem. This has not yet been done.

The contribution $e_{1,0} = 1/6$ has been worked out mathematically in the appendix to [2] using degenerate graphs as degenerate instantons; there the obstruction analysis is simpler than in the problem just outlined. In this case as well as the case of genus 0 multiple covers considered in Section 21 the use of stable maps seems to complicate the analysis.

In [3], the values $e_{2,0} = 1/240$ and $e_{2,1} = 0$ were calculated by the method of undetermined coefficients in a calculation of F_2 for the quintic threefold with the aid of mirror symmetry.

There is a bit of speculation worth making here. Preliminary analysis makes it conceivable that the obstruction problem on $C \times \overline{M}_{g'',1}$ factors nat-

urally into a complicated term on $\overline{M_{g'',1}}$ and the tangent bundle of C . This would imply that for $g - g' \geq 2$

$$e_{g,g'} = (1 - g')e_{g-g',0} \quad (23)$$

Note that this is consistent with $e_{2,1} = 0$, and would imply that $e_{g,1} = 0$ for arbitrary g (this last equality is also consistent with the reasoning in [3]). The first non-trivial check of this speculation would be to see if $e_{4,2} = -1/240$.

5 Conclusions

It appears that any natural method in algebraic geometry for compactifying instanton moduli spaces and performing the path integral over fermion zero modes gives answers which agree with calculations done by mirror symmetry. Deformation theory provides a natural framework for the analysis. The incorporation of degenerate instantons appears essential in any formulation of the theory.

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